Back-Paper Exam ISI Bangalore December 2023

MMath Algebra I 100 Points

**Notes.**

(a) You may freely use any result proved in class unless you have been asked to prove the same. Use your judgement. All other steps must be justified.

(b) We use $\mathbb{N}$ = natural numbers, $\mathbb{Z}$ = integers, $\mathbb{Q}$ = rational numbers, $\mathbb{R}$ = real numbers, $\mathbb{C}$ = complex numbers.

(c) There are a total of **108** points in this paper. You will be awarded a maximum of **100** points.

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1. [18 Points] Let $A$ be a ring. Consider the natural map between polynomial rings $\phi\colon A[x,y] \to A[t]$ induced by sending $x$ to $t^2$ and $y$ to $t^3$. Prove that the kernel of $\phi$ is a principal ideal and find its generator.

2. [18 Points] Let $\mathfrak{m}$ be a maximal ideal in a ring $A$. Prove that for any integer $k > 0$, every element of $A/\mathfrak{m}^k$ is either a unit (multiplicatively invertible) or a nilpotent element.

3. [18 Points] Define what it means for a module $M$ over a ring $R$ to be a noetherian module. Give an example of a ring $R$ and a finitely generated module $M$ such that $M$ is not a noetherian module. Prove that for $R$-modules $M_1, M_2$, we have $M_1 \oplus M_2$ is noetherian if and only if $M_1$ and $M_2$ are noetherian.

4. [18 Points] Prove that $R = \mathbb{Z}[\sqrt{2}]$ is a principal ideal domain.

5. [18 Points] Let $R$ be a unique factorisation domain. Let $a$ and $b \neq 0$ be elements in $R$ having no common factors. Prove that the kernel of the natural map $R[x] \to R[a/b]$ induced by sending $x$ to $a/b$ is generated by $(bx - a)$.

6. [18 Points] Classify up to isomorphism, all $\mathbb{Z}$-modules $M$, such that $M$ has $10^4$ elements and the annihilator of $M$ is $20\mathbb{Z}$.